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Unsteady Flows of Low Concentrated Aqueous Polymer Solutions Through a Planar Channel with Wall Slip

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ABSTRACT

The objective of this paper is to investigate unsteady flows of a low concentrated aqueous polymer solution through a planar channel under Navier-type slip boundary conditions. We obtain analytical formulas for the calculation of the time-dependent velocity field. We also show the existence of a unique solution for the corresponding initial-boundary value problem in the generalized formulation.

Key words: polymer solution, slip boundary condition, channel flow, time-dependent velocity field, analytical solution

INTRODUCTION

Polymers and their solutions are widely used in many technological applications. The calculation of the velocity field of polymeric fluids is a complex and important problem. Due to diversity the nature of polymeric fluids, several models have been suggested in the literature. We investigate unsteady flows of low concentrated aqueous polymer solutions [1–3]. Among all motions used, the flow between two fixed parallel plates plays a fundamental role. This is one of the few situations in which analytic expressions for the flow field are available. The exact solutions of hydrodynamic equations are very important in practise. They make it possible to estimate the applicability area of hydrodynamic models. Exact solutions are also indispensable for testing the relevant numerical and approximation methods.

We are looking for analytical solutions of the initial-boundary value problem that describe unsteady flows of low concentrated aqueous polymer solutions through a plane channel. For the best of our knowledge, this problem has not been solved. The exact solution for the steady flows of polymeric fluids was given by Hron, Le Roux, Málek, and Rajagopal [4]. Some other results were obtained in [5–7].

It is well known that boundary conditions play a key role in the solution determination (see e.g. [8]). The assumption that a fluid adheres to a solid boundary is one of the central tenets of fluid dynamics. However, that conception is not satisfactory for all types of media. It has been widely accepted that a large class of polymeric fluids slip or stick-slip on solid boundaries [9, 10]. In order to construct solutions, we use Navier's slip boundary conditions (see [11] for the original reference) which imply that the slip velocity depends on the shear stress. Due to the special properties of the equations for aqueous polymer solutions motion – the solvability for the initial-boundary value problem in the classical formulation cannot be guaranteed, for any initial condition. In that case, the concept of a generalized solution is useful (see [12]). We provide the generalized formulation and prove the existence and uniqueness of generalized solutions. We also obtain analytical expressions for the time-dependent solutions.

PROBLEM FORMULATION

We consider a low concentrated aqueous polymers solution filling a region between two parallel (infinite) plates z=0 and z=h. Flow is driven by constant pressure gradient. This flow is described by a velocity field with only one nonzero component, v(z,t), directed along the channel and depending only on the coordinates z and on time t. In this case, the motion equation for the aqueous polymer solutions is reduced to

$$\frac{\partial \mathrm{v}}{\partial t} = \nu \frac{\partial^2 \mathrm{v}}{\partial z^2} + \kappa \frac{\partial^3 \mathrm{v}}{\partial t \partial z^2} + \xi_0, \qquad 0 < z < h, \; 0 < t < T, \tag{1}$$

where ν is the coefficient of viscosity, κ/ν is the time of strain relaxation, and $-\xi_0$ is the pressure gradient.

We use the Navier's slip boundary condition that in our case can be written as

$$k\mathrm{v} = \nu \frac{\partial \mathrm{v}}{\partial z} \quad on \ z = 0 \tag{2}$$

$$-k\mathrm{v} = \nu \frac{\partial \mathrm{v}}{\partial z} \quad on \ z = h \tag{3}$$

where the slip coefficient k is a given constant, k>0.
We assume that the velocity field is given at time t=0

$$\mathrm{v}(z,0) = u_0(z), \ \ 0 \le z \le h \tag{4}$$

and the function $u_0 \in C^2[0,h]$ satisfies the compatibility conditions

$$ku_0(0) = \nu u_0'(0),$$

$$-ku_0(h) = \nu u_0'(h).$$

The aim of this paper is to investigate analytical solutions of initial-boundary value problem (1)–(4).

RESULTS

We begin with the following proposition.

Proposition 1

Problem (1-4) *cannot have two different classical solutions.*

Proof

Suppose v_1 and v_2 are classical solutions of problem (1-4), and u= v_1-v_2. Show that u=0. It is easy to see that

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} + \kappa \frac{\partial^3 u}{\partial t \partial z^2}, \quad 0 < z < h, 0 < t < T, \tag{5}$$

$$ku = \nu \frac{\partial u}{\partial z} \ on \ z = 0, \tag{6}$$

$$-ku = \nu \frac{\partial u}{\partial z} \ on \ z = h, \tag{7}$$

$$u(z,0) = 0, \quad 0 \le z \le h. \tag{8}$$

Multiply equality (5) by $u(z,t)$. By integrating the obtained equality with respect to z from 0 to h, we obtain

$$\int_0^h \frac{\partial u}{\partial t} u \, dz = \nu \int_0^h \frac{\partial^2 u}{\partial z^2} u \, dz + \kappa \int_0^h \frac{\partial^3 u}{\partial t \partial z^2} u \, dz$$

Using the formula $(f^2)' = 2ff'$ and the rule of integration by parts, we get

$$\frac{1}{2}\int_0^h \frac{\partial}{\partial t}\left(u^2\right)dz = -\nu \int_0^h \left(\frac{\partial u}{\partial z}\right)^2 dz + \nu \frac{\partial u}{\partial z}(h,t)u(h,t) - \nu \frac{\partial u}{\partial z}(0,t)u(0,t) - \frac{\kappa}{2}\int_0^h \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}\right)^2 dz$$

$$+ \kappa \frac{\partial^2 u}{\partial t \partial z}(h,t)u(h,t) - \kappa \frac{\partial^2 u}{\partial t \partial z}(0,t)u(0,t)$$

Taking into account (6), (7), we obtain

$$\frac{1}{2}\int_0^h \frac{\partial}{\partial t}\left(u^2\right)dz = -\nu \int_0^h \left(\frac{\partial u}{\partial z}\right)^2 dz - ku^2(h,t) - ku^2(0,t) - \frac{\kappa}{2}\int_0^h \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}\right)^2 dz$$

$$-\frac{\kappa k}{2\nu}\frac{\partial u^2}{\partial t}(h,t) - \frac{\kappa k}{2\nu}\frac{\partial u^2}{\partial t}(0,t)$$

Integrating the obtained equality with respect to t from 0 to τ, we obtain

$$\int_0^h u^2(z,\tau)dz \le 0.$$

This yields that $u \equiv 0$.

The solvability for initial-boundary value problem (1-4) in the classical formulation cannot be guaranteed. We provide the generalized formulation and construct the generalized solutions.

First, we describe the concept of a generalized solution.

Definition

We shall say that the function v is a *generalized solution* to (1-4) if there exists a sequence $\{\mathrm{v}_m\}$ such that

(i) the sequence $\{\mathrm{v}_m\}$ is uniformly convergent with limit v,

(ii) the function v_m is a classical solution to (1-3) and $\mathrm{v}_m(\cdot,0)=u_{0m}(\cdot)$ for every natural number m,

(iii) the sequence $\{u_{0m}\}$ is uniformly convergent with limit u_0,

(iv) $(u_{0m})' \to (u_0)'$ in the Hilbert space L_2 (0, h)

Remark 1
If a function v is a classical solution to (1-4), then v is a generalized solution to (1-4).

Remark 2
Generalized solution has a clear physical meaning. It provides the limit for a sequence of classical solutions for the system with small perturbations in the initial moment when the perturbations uniformly converge to zero.

Proposition 2
Problem (1-4) has a unique generalized solution. The following formula can be used to calculate the generalized solution

$$\mathrm{v}(z,t)=\mathrm{v}_0(z)+\sum_{m=1}^{\infty}C_m\left(\frac{h}{2}+\frac{\nu kh^2}{\mu_m^2\nu^2+k^2h^2}\right)^{-1}\exp\left(-\frac{\mu_m^2\nu}{h^2+\kappa\mu_m^2}t\right)\sin\left(\frac{\mu_m}{h}z+\theta_m\right),$$

Where $$\mathrm{v}_0(z)=-\frac{\xi_0}{2\nu}z^2+\frac{\xi_0 h}{2\nu}z+\frac{\xi_0 h}{2k},$$

$$C_m=\int_0^h(u_0(z)+\mathrm{v}_0(z))\sin\left(\frac{\mu_m}{h}z+\theta_m\right)dz,\quad \theta_m=\operatorname{arccot}\left(\frac{\nu\mu_m}{kh}\right)$$

the numbers μ_m are positive roots of the equation

$$\cot\mu=\frac{\nu^2\mu^2-k^2h^2}{2\mu\nu kh}.$$

Proof
To find the generalized solution we use the Fourier method. We introduce a new unknown

$$\mathrm{v}(z,t)=\tilde{\mathrm{v}}(z,t)-\mathrm{v}_0(z),$$

where the function v_0 is a solution of the corresponding stationary boundary value problem. As a result, we get the following initial-boundary value problem

$$\frac{\partial\tilde{\mathrm{v}}}{\partial t}=\nu\frac{\partial^2\tilde{\mathrm{v}}}{\partial z^2}+\kappa\frac{\partial^3\tilde{\mathrm{v}}}{\partial t\partial z^2},\ 0<z<h,\ 0<t<T, \tag{9}$$

$$k\tilde{\mathrm{v}}=\nu\frac{\partial\tilde{\mathrm{v}}}{\partial z}\quad on\ z=0, \tag{10}$$

$$-k\tilde{\mathrm{v}}=\nu\frac{\partial\tilde{\mathrm{v}}}{\partial z}\quad on\ z=h, \tag{11}$$

$$\tilde{\mathrm{v}}(z,0)=u_0(z)+\mathrm{v}_0(z),\ 0\le z\le h. \tag{12}$$

First, we will find nontrivial solutions of problem (9)–(11) in the form

$$\tilde{\mathrm{v}}(z,t)=\varphi(t)q(z).$$

From equation (9) it follows that

$$\varphi'(t)\big(q(z)-\kappa q''(z)\big)=\nu\varphi(t)q''(z). \tag{13}$$

We are interested in the case when

$$q-\kappa q''\neq 0.$$

Let z_0 be a number such that

$$q(z_0)-\kappa q''(z_0)\neq 0.$$

Let us denote

$$\lambda=\frac{q''(z_0)}{q(z_0)-\kappa q''(z_0)}.$$

From equation (13) it follows that

$$\varphi'(t)=\lambda\nu\varphi(t). \tag{14}$$

Using (13) and (14), we get

$$\lambda\big(q(z)-\kappa q''(z)\big)=q''(z)$$

Thus,

$$q''(z)-\frac{\lambda}{1+\lambda\kappa}q(z)=0 \tag{15}$$

Furthermore, in order to satisfy the boundary conditions (10) and (11) the conditions

$$-\nu q'(0)+kq(0)=0, \tag{16}$$

$$\nu q'(h)+kq(h)=0. \tag{17}$$

must be satisfied.

The problem (15-17) is one of the simplest cases of the Sturm–Liouville problem (see e. g. [13]). In our case, the eigen values are the numbers $(\mu_m/h)^2$ (m=1,2, …), where the numbers μ_m are positive roots of the equation

$$\cot\mu = \frac{\nu^2\mu^2 - k^2h^2}{2\mu\nu kh}.$$

Normalized in the Hilbert space $L_2(0, h)$ the eigenfunctions are

$$q_m(z) = \left(\frac{h}{2} + \frac{\nu kh^2}{\mu_m^2\nu^2 + k^2h^2}\right)^{-\frac{1}{2}} \sin\left(\frac{\mu_m}{h}z + \theta_m\right), \quad \theta_m = \operatorname{arccot}\left(\frac{\nu\mu_m}{kh}\right).$$

The parameter λ is determined by the condition

$$-\frac{\lambda}{1+\lambda\kappa} = \left(\frac{\mu_m}{h}\right)^2.$$

Therefore,

$$\lambda = -\frac{\mu_m^2}{h^2 + \kappa\mu_m^2}.$$

We substitute the values λ in (14) and find the appropriate solutions:

$$\varphi_m(t) = C\exp\left(-\frac{\mu_m^2\nu}{h^2+\kappa\mu_m^2}t\right), \quad C = const, \ m = 1,2,...$$

Thus, we have the sequence of solutions of (9-11)

$$\tilde{\mathrm{v}}_m(z,t) = \left(\frac{h}{2} + \frac{\nu kh^2}{\mu_m^2\nu^2 + k^2h^2}\right)^{-\frac{1}{2}} \exp\left(-\frac{\mu_m^2\nu}{h^2+\kappa\mu_m^2}t\right)\sin\left(\frac{\mu_m}{h}z + \theta_m\right), \ m = 1,2,...$$

Using the functions $\tilde{\mathrm{v}}_m$, we construct the generalized solution of problem (9-12).

Consider a uniformly convergent series

$$\tilde{\mathrm{v}}(z,t) = \sum_{m=1}^{\infty} A_m \tilde{\mathrm{v}}_m(z,t),$$

where

$$A_m = \int_0^h (u_0(z) + \mathrm{v}_0(z))\tilde{\mathrm{v}}_m(z,0)dz.$$

Let us show that the function $\tilde{\mathrm{v}}$ is a generalized solution of problem (9-12).

Set

$$\tilde{\mathrm{v}}^{(m)}(z,t) = \sum_{i=1}^{m} A_i \tilde{\mathrm{v}}_i(z,t).$$

It is clear that $\tilde{\mathrm{v}}^{(m)}$ is a classical solution of problem (9)–(11). By the Steklov theorem (see e. g. [13]), the sequence $\tilde{\mathrm{v}}^{(m)}(\cdot,0)$ converges uniformly to $u_0 + \mathrm{v}_0$. Furthermore, it is easily shown that the sequence $(\tilde{\mathrm{v}}^{(m)}(\cdot,0))'$ converges $(u_0 + \mathrm{v}_0)'$ in space $L_2(0, h)$ as $m\to\infty$. Thus all the conditions of the definition of a generalized solution are valid. The uniqueness of the generalized solution is proved by an argument similar to the proof of Proposition 1. This concludes the proof.

CONCLUSIONS

We have considered unsteady flows of a low concentrated aqueous polymer solution through a planar channel under Navier-type slip boundary conditions. We have defined a generalized solution for the corresponding initial-boundary value problem. Using the theory of Sturm-Liouville problem, we obtain the analytic generalized solutions for the time-dependent velocity field. The results can be used for further study of complex unsteady flows of aqueous polymer solutions, including a pulsating planar flow, which is important in practise.

REFERENCES

[1] VA Pavlovskii, To a Problem on Theoretical Exposition of Weak Aqueous Solutions of Polymers, *DAN USSR*, **1971**, 200, 809-812.

[2] AP Oskolkov, On the Uniqueness and Global Solvability of Boundary Value Problems for the Equations of Motion of Aqueous Polymer Solutions, *Boundary Value Problems in Mathematical Physics and Related Issues of Function Theory*, **1973** , 7, 98-136.

[3] AP Oskolkov, The Initial Boundary-Value Problem with a Free Surface Condition for the Penalized Equations of Aqueous Solutions of Polymers, *Journal of Mathematical Sciences*, **1997**, 83, 320-326.

[4] J Hron, C Le Roux, J Málek and KR Rajagopal, Flows of Incompressible Fluids subject to Navier's slip on the Boundary, *Computers and Mathematics with Applications*, **2008**, 56, 2128-2143.
[5] M Khan, S Nadeem, T Hayat and AM Siddiqui, Unsteady Motions of a Generalized Second-grade Fluid, *Mathematical and Computer Modelling*, **2005**, 41, 629-637.
[6] ES Baranovskii, On Steady Motion of Viscoelastic Fluid of Oldroyd Type, *Sbornik: Mathematics*, **2014**, 205, 763-776.
[7] ES Baranovskii, Flows of a Polymer Fluid in Domain with Impermeable Boundaries, *Computational Mathematics and Mathematical Physics*, **2014**, 54, 1589-1596.
[8] KR Rajagopal, On Some Unresolved Issues in Non-linear Fuid Dynamics, *Russian Math. Surveys*, **2003**, 58, 319-330.
[9] M Denn, Extrusion Instabilities and Wall Slip, *Annu. Rev. Fluid Mech*, **2001**, 33, 265-287.
[10] T Hayat, Masood Khan and M Ayub, On Non-linear Fows with Slip Boundary Condition, *Z. Angew. Math. Phys.* **2005**, 56, 1012-1029.
[11] CLMH Navier, Memoire sur les lois du mouvement des fluides, *Mem. Acad. R. Sci.*, **1823**, 6, 389-416.
[12] SL Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, 3rd ed., Providence: Amer. Math. Soc., **1991**.
[13] VS Vladimirov, *Equations of Mathematical Physics*, New York: Marcel Dekker, **1971**.