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# Finite Element Solver for 2D Linear Elasticity Beam

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**ABSTRACT**

*The goal of this article is to know how much increases the stiffness matrix conditioning of steel cantilever beam by developing a finite element code for 2D linear elasticity, using a triangular finite element in the beam modeling with MATLAB [1].*

**Key words**: Finite element, Triangular element, Weak formulation, Eigenvalues, Residue of iteration

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## INTRODUCTION

In this article the project is to study the deformation of a steel cantilever beam ; embedded on one side $\Gamma_D \subset \partial\Omega$ and free in another $\Gamma_N = \frac{\partial\Omega}{\Gamma_D}$ which is subjected to in each point to the action of body forces $x \in \Omega$ in his free end side as represented in Fig. 1.

The external forces and internal forces at each point of Ω requires that for each component. i = {1, 2} of the vector forces we have:

$$\sum_j \frac{\partial \sigma_{ij}(u)}{\partial x_j} + f_i = 0 \quad \text{in } \Omega \tag{1}$$

$$u = 0 \qquad \text{on } \Gamma_D \tag{2}$$

$$\sigma_{ij} n_j = g_i \qquad \text{on } \Gamma_N \tag{3}$$

where $\sigma_{ij}(u), i, j = \{1,2\}, \sigma_{ij} = \sigma_{ji}$ is the Stress tensor (matrix) which is, proportional to deformation of the solid. According to Hooke's law, we have the following relation between the stress tensor $\sigma_{ij}$ and the deformation tensor $\varepsilon_{ij}$

$$\sigma_{ij}(u) = \frac{E}{(1+\upsilon)}\left(\sum_k \frac{\upsilon}{(1-2\upsilon)} \varepsilon_{kk}\delta_{ij} + \varepsilon_{ij}\right) \tag{4}$$

where the young's modulus E = $2.10^{11}$ [ PA], Poisson ratio υ for steel is equal to 0.3. The strain tensor $\varepsilon_{ij}(u)$ is expressed in term of displacement u(x) as following :

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{5}$$

### Weak Formulation of the Problem

According to the equation (1) , we have $\sum_j \frac{\partial \sigma_{ij}(u)}{\partial x_j} + f_i = 0$ By multiplying by v (function test), and then summing for each i we obtain[2-4]:

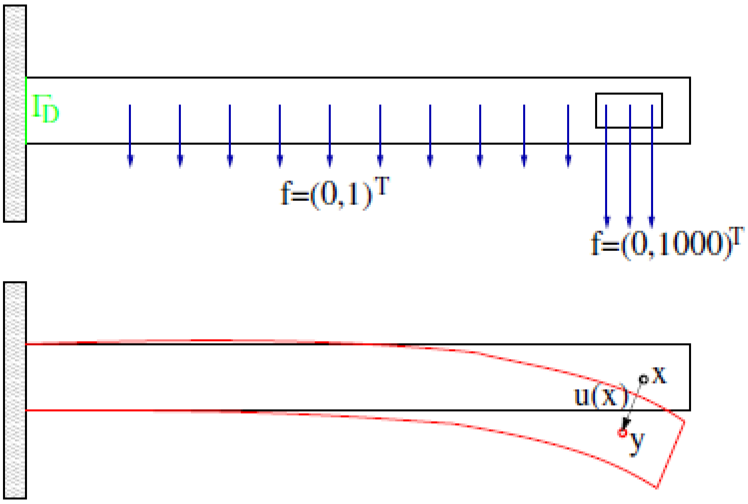


**Fig. 1 Geometry of the beam**

$$\sum_{i,j} \frac{\partial \sigma_{ij}(u)}{\partial x_j} v_i + \sum_i f_i v_i = 0 \tag{6}$$

$$\sum_{i,j} \frac{\partial \sigma_{ij}(u)}{\partial x_j} v_i = -\sum_i f_i v_i$$

By integrating over Ω

$$\int_\Omega \sum_{i,j} \frac{\partial \sigma_{ij}(u)}{\partial x_j} v_i = -\int_\Omega \sum_i f_i v_i$$

According to the divergence theorem

$$\int_\Omega \sum_{i,j} \frac{\partial \sigma_{ij}(u)}{\partial x_j} v_i = -\int_\Omega \sum_{i,j} \sigma_{ij}(u) \frac{\partial u_i}{\partial x_j} d\Omega + \int_\Gamma \sum_{i,j} \sigma_{ij}(u) v_i n_j d\Gamma$$

knowing that

$$\int_\Gamma \sum_{i,j} \sigma_{ij}(u) v_i n_j d\Gamma = \int_{\Gamma_D} \sum_{i,j} \sigma_{ij}(u) v_i n_j d\Gamma + \int_{\Gamma_N} \sum_{i,j} \sigma_{ij}(u) v_i n_j d\Gamma$$

because $v_i = 0$ in $\Gamma_D$ and $\sigma_{ij} n_i = g_i = 0$ in $\Gamma_N$

using symmetry $\sigma_{ij} = \sigma_{ji}$

$$-\int_\Omega \sum_{i,j} \sigma_{ij}(u) \frac{\partial v_i}{\partial x_j} d\Omega = -\int_\Omega \left( \sum_{i,j} \frac{1}{2} \sigma_{ij}(u) \frac{\partial v_i}{\partial x_j} + \sum_{i,j} \frac{1}{2} \sigma_{ij}(u) \frac{\partial v_j}{\partial x_i} \right) d\Omega$$
$$= -\int_\Omega \sum_{i,j} \sigma_{ij}(u) \varepsilon_{ij}(v) d\Omega \tag{7}$$

with $\varepsilon_{ij}(v) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$

finally, we get the weak formulation of the problem

$$\int_\Omega \sum_{i,j} \sigma_{ij}(u) \varepsilon_{ij}(v) d\Omega = \int_\Omega \sum_i f_i v_i d\Omega \tag{8}$$

## DISCRETIZATION OF THE PROBLEM USING FINITE ELEMENT METHOD

The triangular element is used for 2D model of beam structure; assuming an element with three nodes i,j,k in this case each node displacement has two components as follows [2-3-4]:

$$u_i = \left(u_{i1}, u_{i2}\right)^t \quad u_j = \left(u_{j1}, u_{j2}\right)^t \quad u_k = \left(u_{k1}, u_{k2}\right)^t \tag{9}$$

From the geometry of the triangle, we can define

$$\begin{cases} \phi_i(x) = a_i + b_i x_1 + c_i x_2 \\ \phi_j(x) = a_j + b_j x_1 + c_j x_2 \\ \phi_k(x) = a_k + b_k x_1 + c_k x_2 \end{cases} \qquad (10)$$

If $l$ is one of the summits of the triangle, we have by definition

$$\begin{cases} \phi_i(x_l) = 1 \quad if \quad l = i \quad otherwise \quad 0 \\ \phi_j(x_l) = 1 \quad if \quad l = j \quad otherwise \quad 0 \\ \phi_k(x_l) = 1 \quad if \quad l = k \quad otherwise \quad 0 \end{cases}$$

we obtain the following linear system:

$$\begin{bmatrix} 1 & x_{j1} & x_{j2} \\ 1 & x_{j1} & x_{j2} \\ 1 & x_{j1} & x_{j2} \end{bmatrix} \begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} = \begin{bmatrix} 1 & x_{j1} & x_{j2} \\ 1 & x_{j1} & x_{j2} \\ 1 & x_{j1} & x_{j2} \end{bmatrix}^{-1}$$

we seek to estimate

$$\varepsilon_{11}(u), \varepsilon_{12}(u), \varepsilon_{22}(u)$$

$$\varepsilon_{11}(u) = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u_1}{\partial x_1}$$

$$u_1 = u_{i1}\phi_i + u_{j1}\phi_j + u_{k1}\phi_k$$

so

$$\varepsilon_{11}(u) = u_{i1}\frac{\partial \phi_i}{\partial x_i} + u_{j1}\frac{\partial \phi_j}{\partial x_j} + u_{k1}\frac{\partial \phi_k}{\partial x_1}$$

$$= u_{i1}b_i + u_{j1}b_j + u_{k1}b_k$$

as well

$$\varepsilon_{22}(u) = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) = \frac{\partial u_2}{\partial x_2}$$

$$u_2 = u_{i2}\phi_i + u_{j2}\phi_j + u_{k2}\phi_k$$

so

$$\varepsilon_{22}(u) = u_{i2}\frac{\partial \phi_i}{\partial x_i} + u_{j2}\frac{\partial \phi_j}{\partial x_j} + u_{k2}\frac{\partial \phi_k}{\partial x_1}$$

$$= u_{i2}b_i + u_{j2}b_j + u_{k2}b_k$$

$$\varepsilon_{12}(u) = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

finally

$$\varepsilon_{12}(u) = \frac{1}{2}(u_{i1}c_i + u_{j1}c_j + u_{k1}c_k + u_{i2}b_i + u_{j2}b_j + u_{k2}b_k)$$

Hence the expression of strain matrix:

$$\begin{bmatrix} \varepsilon_{11}(u) \\ \varepsilon_{12}(u) \\ \varepsilon_{22}(u) \end{bmatrix} = \begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ \frac{1}{2}c_i & \frac{1}{2}b_i & \frac{1}{2}c_j & \frac{1}{2}b_j & \frac{1}{2}c_k & \frac{1}{2}b_k \\ 0 & c_i & 0 & c_j & 0 & c_k \end{bmatrix} \begin{bmatrix} u_{i1} \\ u_{i2} \\ u_{j1} \\ u_{j2} \\ u_{k1} \\ u_{k2} \end{bmatrix} \qquad (11)$$

Using the relation (4) the constitutive matrix is written

$$\begin{bmatrix}\sigma_{11}(u)\\ \sigma_{12}(u)\\ \sigma_{22}(u)\end{bmatrix}=\frac{E}{(1+\nu)}\begin{bmatrix}\frac{\nu}{(1-2\nu)}+1 & 0 & \frac{\nu}{(1-2\nu)}\\ 0 & 1 & 0\\ \frac{\nu}{(1-2\nu)} & 0 & \frac{\nu}{(1-2\nu)}+1\end{bmatrix}\begin{bmatrix}\varepsilon_{11}(u)\\ \varepsilon_{12}(u)\\ \varepsilon_{22}(u)\end{bmatrix} \tag{12}$$

## DISCRETIZATION AND APPROXIMATION OF THE SECOND MEMBER

The goal of this section is to compute the stiffness matrix for Laplace, with homogeneous Neumann conditions [5]:

$$\begin{aligned}\int_{\Omega}\sum_{i} f_i v_i \, d\Omega &= \sum_{K}\int_{K} f_1\left|Kv_1\right|K + f_2\left|Kv_2\right|K \, d\Omega\\ &= \sum_{K}\int_{K}\left(f_{i1}\phi_i + f_{j1}\phi_j + f_{k1}\phi_k\right)\bullet\left(v_{i1}\phi_i + v_{j1}\phi_j + v_{k1}\phi_k\right)d\Omega\\ &+ \sum_{K}\int_{K}\left(f_{i2}\phi_i + f_{j2}\phi_j + f_{k2}\phi_k\right)\bullet\left(v_{i2}\phi_i + v_{j2}\phi_j + v_{k2}\phi_k\right)d\Omega\end{aligned} \tag{13}$$

$$K\begin{bmatrix}u_{11}\\ u_{12}\\ u_{21}\\ u_{22}\\ \vdots\\ u_{N1}\\ u_{N2}\end{bmatrix}=\begin{bmatrix}f_{11}\\ f_{12}\\ f_{21}\\ f_{22}\\ \vdots\\ f_{N1}\\ f_{N2}\end{bmatrix}$$

Where for a node i

$$f_{i1}=\int_{\Gamma} f_1(x_1,x_2)\phi_i(x_1,x_2)$$

$$f_{i1}=\int_{\Gamma} f_2(x_1,x_2)\phi_i(x_1,x_2)$$

In our case $f_1$ is zero so we can make the following approximation

$$\begin{aligned}\int_{\Omega} f_2(x_1,x_2)\phi_i(x_1,x_2) &= \sum_{K,i\in K}\int_{K} f_2\left(K\right)\phi_i \, d\Omega\\ &\approx \sum_{K,i\in K}\int_{K}\frac{1}{3}|K|\, f_2\left(x_i,y_i\right)\\ &= f_2\left(x_i,y_i\right)\sum_{K,i\in K}\frac{1}{3}|K|\end{aligned} \tag{14}$$

With $\sum_{K,i\in K}\frac{1}{3}|K| = CellArea$

## NUMERICAL RESULTS AND ITERATIVE METHODS

The beam considered is 20 times longer than wide see Fig. 2. The stopping criterion of the iterative method is [6-7]:

$$|R_i| / |R_0| \prec 10^{-6}$$

$R_i$ is the residue at i^th iteration, $R_0$ the residue before the iteration, m represent a number of node in X direction, n represent a number of node in Y direction, $\lambda_{max}$ represents the maximum eigenvalue and $\lambda_{min}$ represents the minimum eigenvalue

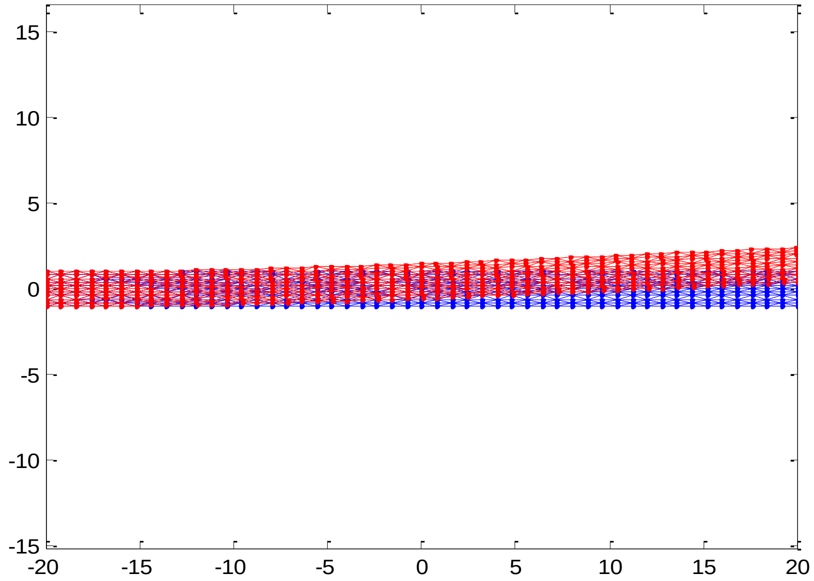


**Fig. 2 Deformation shape of the beam element**

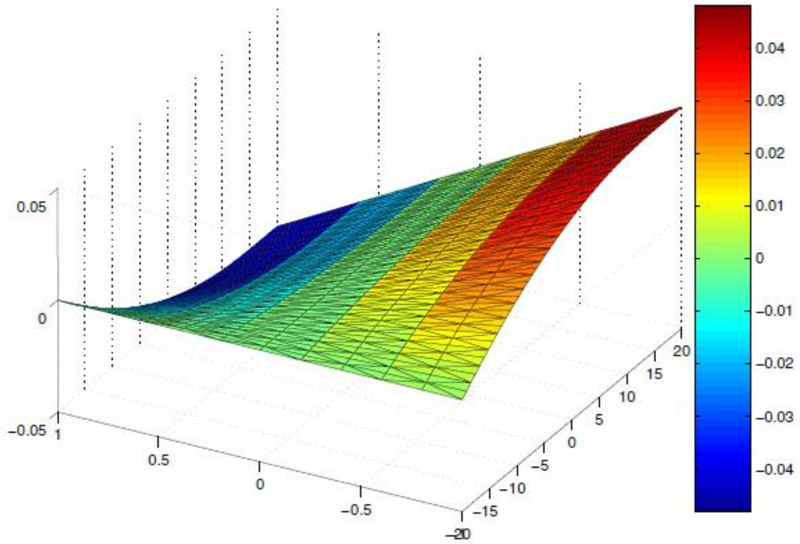


**Fig. 3 Deformation in x direction**

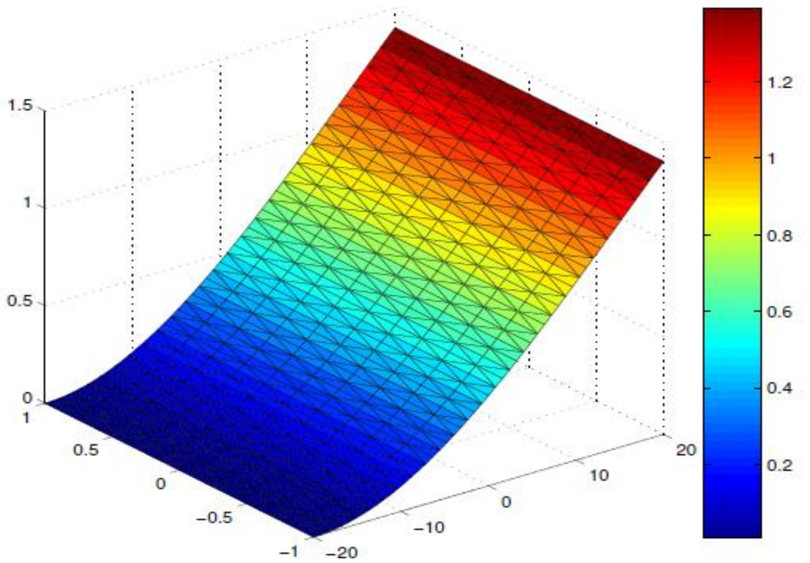


**Fig. 4 Deformation in y direction**

**Table -l Eigenvalues Results for Different Meshes**

| m | 10 | 20 | 40 | 80 | 160 |
|---|---|---|---|---|---|
| n | 4 | 8 | 16 | 32 | 64 |
| Number of iteration | 205 | 643 | 1729 | 3736 | 7981 |
| $\lambda_{max} / \lambda_{min}$ | $5.2x10^6$ | $3.78x10^7$ | $1.89x10^8$ | $7.98x10^8$ | $3.19x10^9$ |

## RESULTS AND DISCUSSION

We notice that the conditioning increases in $O(1 / h^2)$ when the discretization step h is small enough. In our example, this corresponds to the m values greater than 40 and n greater than 16 (see table 1). Our global stiffness matrix K is not well conditioned; it is hardly reversible. Many iterations are required to have a sufficiently negligible residue and find an acceptable solution. We also note that the conditioning of global matrix increases as the mesh is refined.

## CONCLUSIONS

The appearance of a second degree of freedom required a general reformulation of the problem. The construction of the elementary matrix is based on the construction of the stress and train tensor. A reformulation of the matrix indices I and J was necessary for vectoring the elementary matrix. The discretization of the second member oblige Cell Area calculation per nodes. The mesh should not be too refining in order to solve the system but enough to get a good approximation of the exact solution.

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