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Composition Formulae for Unified Fractional Integral Operators

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ABSTRACT

In the present Paper, we derive three new and interesting expressions for the composition of two most general fractional integral operators whose kernels involve the product of a Appell Polynomial, Fox H-function and S-Generalized Gauss's Hypergeometric Function. The operators of our study are quite general in nature and may be considered as extensions of a number of simpler fractional integral operators studied from time to time by several authors. By suitable specializing the coefficients and the parameters in these functions we can get a large number of (new and known) interesting expressions for the composition of fractional integral operators involving simpler special functions. The results obtained by Erdélyi [2], Goyal and Jain [11] follow as simple cases of our composition formulae.

Key words: Unified Fractional integral operators, Appell Polynomial, S-Generalized Gauss's Hypergeometric Function, Fox H-function.

INTRODUCTION

Fox H-Function

A single Mellin-Barnes contour integral, occurring in the present work, is now popularly known as the H-function of Charles Fox (1897-1977). It will be defined and represented here in the following manner (see, for example, [8]):

$$H_{P,Q}^{M,N}[z]=H_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_P,\alpha_P)\\(b_1,\beta_1),\ldots,(b_Q,\beta_Q)\end{matrix}\right.\right]=H_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_P,\alpha_P)\\(b_Q,\beta_Q)\end{matrix}\right.\right] \quad =\frac{1}{2\pi i}\int_L \theta(\xi)z^{\xi}\,d\xi, \tag{1}$$

where $i=\sqrt{-1}, z\in C\setminus\{0\}$, C being the set of complex numbers

$$\theta(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=M+1}^{Q}\Gamma(1-b_j+\beta_j\xi)\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \tag{2}$$

And $\quad 1\le M\le Q$ and $0\le N\le P$ $\quad (M, Q \in N = \{1, 2, 3,...\};\ N, P \in N_0= N\cup\{0\})$ (3)

an empty product being interpreted to be 1. Here L is a Mellin - Barnes type contour in the complex ξ-plane with appropriate indentations in order to separate the two sets of poles of the integrand $\theta(\xi)$ [1] and [8].

Multivariable H-Function

The Multivariable H-Function is defined and represented in the following manner [8]:

$$H_{p,q:p_1,q_1;\ldots;p_r,q_r}^{0,n:m_1,n_1;\ldots;m_r,n_r}\left(\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)})_{1,p}:(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,q}:(d_j^{(1)},\delta_j^{(1)})_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right.\right)$$

$$=\frac{1}{(2\pi\omega)^{r}}\int_{L_1}\cdots\int_{L_r}\phi(\xi_1,\ldots,\xi_r)\prod_{i=1}^{r}\{\theta_i(\xi_i)z_i^{\xi_i}\}d\xi_1\ldots d\xi_r \qquad \text{where } \omega=\sqrt{-1} \tag{4}$$

$$\phi(\xi_1,\ldots,\xi_r)=\frac{\prod_{j=1}^{n}\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i)}{\prod_{j=1}^{q}\Gamma(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\xi_i)\prod_{j=n+1}^{p}\Gamma(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i)},$$

$$\theta_i(\xi_i)=\frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}\xi_i)\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i)}{\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}\xi_i)\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi)}, \qquad (i=1,2,\ldots,r) \tag{5}$$

All the Greek letters occurring on the left and side of (1.4) are assumed to be positive real numbers for standardization purposes. The definition of the multivariable H-function will however be meaningful even if some of these quantities are zero. The details about the nature of contour $L_1,\ldots,L_r$, conditions of convergence of the integral given by (1.4). Throughout the paper it is assumed that this function always satisfied its appropriate conditions of convergence [8].

S-Generalized Gauss's Hypergeometric Function

The S-generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z)$ introduced and defined by Srivastava et al. [9] is represented in the following manner:

$$F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \qquad (|z|<1) \tag{6}$$

Provided that (Re (p) ≥ 0; minRe{α, β, τ, μ } > 0: Re (c) > Re (b) > 0)

where the S-generalized Beta function $B_p^{(\alpha,\beta;\gamma,\tau)}(x,y)$ was introduced and defined by Srivastava et al [9]:

$$B_p^{(\alpha,\beta;\tau,\mu)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^{\tau}(1-t)^{\mu}}\right)dt \tag{7}$$

provided that (Re (p) ≥ 0; minRe{x, y, α, β, τ, μ } > 0)

and $(\lambda)_n$ denotes the pochhammer symbol defined (for $\lambda\in C$) by (see [5];see also [4]):

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1 & if \quad n=0\\ \lambda(\lambda+1)\ldots(\lambda+n-1), if & n=1,2...\end{cases} \tag{8}$$

provided that the Gamma quotient exists (see, for details, [6] and [7]).

Appell Polynomial

The Appell Polynomial introduced and defined by [10] is represented in the following manner:

$$A_n(z)=\sum_{k=0}^{n}a_{n-k}\frac{z^k}{k!} \qquad n = 0, 1, 2, \ldots. \tag{9}$$

Where a_{n-k} is the complex coefficient $a_0\neq 0$.

Fractional Integral Operators

We study two unified fractional integral operators involving the Appell Polynomial, Fox H-function and S-Generalized Gauss's Hypergeometric Function having general arguments

$$I_x^{\nu,\lambda}\{A_n,H,F_p;f(t)\}=x^{-\nu-\lambda-1}\int_0^x t^{\nu}(x-t)^{\lambda}A_n\left(z_1\left(\frac{t}{x}\right)^{\nu_1}\left(1-\frac{t}{x}\right)^{\lambda_1}\right)$$

$$H_{p,q}^{m,n}\left[z_2\left(1-\frac{t}{x}\right)^{\lambda_2}\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{matrix}\right.\right]F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(1-\frac{t}{x}\right)^{\lambda_3}\right]f(t)dt \tag{10}$$

Provided that

$$\min_{1\le j\le m} \operatorname{Re}\left(\nu+\varsigma+1, \lambda+\lambda_2\left(\frac{b_j}{\beta_j}\right)+1\right)>0 \text{ and } \min\{\nu_1,\lambda_1,\lambda_3\}\ge 0 \tag{11}$$

$$J_x^{\nu,\lambda}\{A_n,H,F_p;f(t)\} == x^{\nu}\int_x^{\infty} t^{-\nu-\lambda-1}(t-x)^{\lambda} A_n\left(z_1\left(\frac{x}{t}\right)^{\nu_1}\left(1-\frac{x}{t}\right)^{\lambda_1}\right) H_{p,q}^{m,n}\left[z_2\left(1-\frac{x}{t}\right)^{\lambda_2}\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{matrix}\right.\right] F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(1-\frac{x}{t}\right)^{\lambda_3}\right] f(t)\,dt \tag{12}$$

Provided that

$$\operatorname{Re}(w_2)>0 \text{ or } \operatorname{Re}(w_2)=0 \text{ and } \min_{1\le j\le m}\operatorname{Re}(\nu-w_1)>0;$$

$$\min_{1\le j\le m} \operatorname{Re}\left(\lambda+\lambda_2\left(\frac{b_j}{\beta_j}\right)+1\right)>0 \text{ and } \min\{\nu_1,\lambda_1,\lambda_3\}\ge 0 \tag{13}$$

where, The operators are defined for $f(t)\in\Lambda, \Lambda$ denotes the class of function f(t) for which

$$f(t)=\begin{cases} O\{|t|^{\zeta}\} &, \max\{|t|\}\to 0\\ O\{|t|^{w_1}e^{-w_2|t|}\} &, \min\{|t|\}\to\infty\end{cases} \tag{14}$$

RESULTS

Composition Formulae For The Fractional Operators

Result 1 $$I_{x_2}^{\nu',\lambda'}\left[A_{n'},H',F_{p'};I_{x_1}^{\nu,\lambda}\{A_n,H,F_p;f(t)\}\right]=\frac{1}{x_2}\int_0^{x_2} G\left(\frac{t}{x_2}\right)f(t)\,dt \tag{15}$$

Where

$$G(X)=\frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(a)\Gamma(a')B(b,c-b)B(b',c'-b')}\sum_{k=0}^{n}\sum_{k'=0}^{n'}\frac{a_{n-k}a_{n'-k'}}{k!k'!}z_1^{k}z_1'^{k'}$$

$$X^{\nu+\nu_1 k}(1-X)^{\lambda+\lambda_1 k+\lambda'+\lambda_1' k'+1}\, H_{5,4:3,1;3,1;P',Q';1,1;P,Q;1,1;0,1}^{0,5:1,2;1,2;M',N';1,1;M,N;1,1;1,0}\left[\begin{matrix}p'^{-1}\\p^{-1}\\z_2'(1-X)\\-z_3'(1-X)\\z_2(1-X)\\-z_3(1-X)\\-(1-X)\end{matrix}\left|\begin{matrix}A^*:C^*\\ \\ \\ \\ \\ \\B^*:D^*\end{matrix}\right.\right] \tag{16}$$

Where

$$\left.\begin{aligned}
A^* &= (-\lambda-\lambda_1 k-\nu-\nu_1 k+\nu'+\nu_1' k';0,0,0,0,\lambda_2,\lambda_3,1),(-\lambda'-\lambda_1' k';0,0,\lambda_2',\lambda_3',0,0,1),\\
&(-\lambda-\lambda_1 k;0,0,0,0,\lambda_2,\lambda_3,0),(1-b';\tau',0,0,1,0,0,0),(1-b;0,\tau,0,0,0,1,0)\\
B^* &= (-\lambda-\lambda_1 k-\nu-\nu_1 k+\nu'+\nu_1' k';0,0,0,0,\lambda_2,\lambda_3,0),(-1-\lambda-\lambda_1 k-\lambda'-\lambda_1' k';0,0,\lambda_2',\lambda_3',\lambda_2,\lambda_3,1),\\
&(1-c';\tau'+\mu',0,0,1,0,0,0),(1-c;0,\tau+\mu,0,0,0,1,0)\\
C^* &= (1,1),(1-c'+b',\mu'),(\beta',1);(1,1),(1-c+b,\mu),(\beta,1);(a_j',\alpha_j')_{1,P'};(1-a',1);(a_j,\alpha_j)_{1,P};(1-a,1);-\\
D^* &= (\alpha',1);(\alpha,1);(b_j',\beta_j')_{1,Q'};(0,1);(b_j,\beta_j)_{1,Q};(0,1);(0,1)
\end{aligned}\right\} \tag{17}$$

and following conditions are satisfied

$$\left.\begin{array}{l} f(t)\in\Lambda \\ \operatorname{Re}\left(\nu'+\nu+\zeta\right)>2;\ \min\limits_{1\le j\le M}\operatorname{Re}\left(\lambda'+\lambda_2'\dfrac{b_j'}{\beta_j'}+\lambda+\lambda_2\dfrac{b_j}{\beta_j}\right)>-2 \\ \min\left(\nu_1,\nu_1',\lambda_1,\lambda_1',\lambda_3,\lambda_3'\right)\ge 0 \end{array}\right\} \tag{18}$$

Result 2

$$J_{x_2}^{\nu',\lambda'}\left[A_{n'},H',F_{p'};J_{x_1}^{\nu,\lambda}\left\{A_n,H,F_p;f(t)\right\}\right]=\int_{x_2}^{\infty}\frac{1}{t}G^*\left(\frac{x_2}{t}\right)f(t)dt \tag{19}$$

Where

$$G(Y)=\frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(a)\Gamma(a')B(b,c-b)B(b',c'-b')}\sum_{k=0}^{n}\sum_{k'=0}^{n'}\frac{a_{n-k}a_{n'-k'}}{k!k'!}z_1^k z_1'^{k'}$$

$$Y^{\nu'+\nu_1'k'}(1-Y)^{\lambda+\lambda_1k+\lambda'+\lambda_1'k'+1}\ H_{5,4:3,1;3,1;P',Q';1,1;P,Q;1,1;0,1}^{0,5:1,2;1,2;M',N';1,1;M,N;1,1;1,0}\left[\begin{array}{c|c} p'^{-1} & P^*:R^* \\ p^{-1} & \\ z_2'(1-Y)^{\lambda_2'} & \\ -z_3'(1-Y)^{\lambda_3'} & \\ z_2(1-Y)^{\lambda_2} & \\ -z_3(1-Y)^{\lambda_3} & \\ -(1-Y) & Q^*:S^* \end{array}\right] \tag{20}$$

Where

$$\left.\begin{array}{l} P^*=(-\lambda'-\lambda_1'k'+\nu+\nu_1k-\nu'-\nu_1'k';0,0,\lambda_2',\lambda_3',0,0,1),(-\lambda'-\lambda_1'k';0,0,\lambda_2',\lambda_3',0,0,0), \\ (-\lambda-\lambda_1k;0,0,0,0,\lambda_2,\lambda_3,1),(1-b';\tau',0,0,1,0,0,0),(1-b;0,\tau,0,0,0,1,0) \\ Q^*=(-\lambda'-\lambda_1'k'+\nu+\nu_1k-\nu';0,0,\lambda_2',\lambda_3',0,0,0),(-1-\lambda-\lambda_1k-\lambda'-\lambda_1'k';0,0,\lambda_2',\lambda_3',\lambda_2,\lambda_3,1), \\ (1-c';\tau'+\mu',0,0,1,0,0,0),(1-c;0,\tau+\mu,0,0,0,1,0) \\ R^*=(1,1),(1-c'+b',\mu'),(\beta',1);(1,1),(1-c+b,\mu),(\beta,1);\left(a_j',\alpha_j'\right)_{1,P'};(1-a',1);\left(a_j,\alpha_j\right)_{1,P};(1-a,1);- \\ S^*=(\alpha',1);(\alpha,1);\left(b_j',\beta_j'\right)_{1,Q'};(0,1);\left(b_j,\beta_j\right)_{1,Q};(0,1);(0,1) \end{array}\right\} \tag{21}$$

and following conditions are satisfied

$$\left.\begin{array}{l} \operatorname{Re}(w_2)>0\ or\ \operatorname{Re}(w_2)=0\ and\ \min\operatorname{Re}\left[(\nu+\nu'-w_1)\right]>0 \\ \min\limits_{1\le j\le M}\operatorname{Re}\left(\lambda'+\lambda_2'\dfrac{b_j'}{\beta_j'}+\lambda+\lambda_2\dfrac{b_j}{\beta_j}\right)>-2 \\ \min\left(\nu_1,\nu_1',\lambda_1,\lambda_1',\lambda_3,\lambda_3'\right)\ge 0 \end{array}\right\} \tag{22}$$

Result 3

$$I_{x_2}^{\nu',\lambda'}\left[A_{n'},H',F_{p'};J_{x_1}^{\nu,\lambda}\left\{A_n,H,F_p;f(t)\right\}\right]=\frac{1}{x_2}\int_0^{x_2}K\left(\frac{t}{x_2}\right)f(t)dt+\int_{x_2}^{\infty}\frac{1}{t}K^*\left(\frac{x_2}{t}\right)f(t)dt \tag{23}$$

where

$$K(T)=\frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(a)\Gamma(a')B(b,c-b)B(b',c'-b')}\sum_{k=0}^{n}\sum_{k'=0}^{n'}\frac{a_{n-k}a_{n'-k'}\Gamma(\nu+\nu'+\nu_1k+\nu_1'k'+1)}{k!k'!}z_1^k z_1'^{k'}$$

$$T^{\nu'+\nu_1'k'}(1-T)^{\lambda+\lambda_1 k+\lambda'+\lambda_1' k'+1}\, H^{0,4:1,2;1,2;M',N';1,1;M,N;1,1;1,0}_{4,4:3,1;3,1;P',Q';1,1;P,Q;1,1;0,1}\left[\begin{matrix} p'^{-1} \\ p^{-1} \\ z_2'(1-T)^{\lambda_2'} \\ -z_3'(1-T)^{\lambda_3'} \\ z_2(1-T)^{\lambda_2} \\ -z_3(1-T)^{\lambda_3} \\ -T \end{matrix}\left|\begin{matrix} P^{**}:R^{**} \\ \\ \\ \\ \\ \\ Q^{**}:S^{**}\end{matrix}\right.\right] \tag{24}$$

$$\left.\begin{aligned}
&P^{**}=(-1-\lambda-\lambda_1 k-\lambda'-\lambda_1' k'-\nu-\nu_1 k-\nu'-\nu_1' k';0,0,\lambda_2',\lambda_3',\lambda_2,\lambda_3,1),(-\lambda-\lambda_1 k;0,0,0,0,\lambda_2,\lambda_3,1),\\
&(1-b';\tau',0,0,1,0,0,0),(1-b;0,\tau,0,0,0,1,0)\\
&Q^{**}=(-1-\lambda-\lambda_1 k-\lambda'-\lambda_1' k'-\nu-\nu_1 k-\nu'-\nu_1' k';0,0,\lambda_2',\lambda_3',\lambda_2,\lambda_3,0),\\
&(-1-\lambda-\lambda_1 k-\nu-\nu_1 k-\nu'-\nu_1' k';0,0,0,0,\lambda_2,\lambda_3,1),(1-c';\tau'+\mu',0,0,1,0,0,0),(1-c;0,\tau+\mu,0,0,0,1,0)\\
&R^{**}=(1,1),(1-c'+b',\mu'),(\beta',1);(1,1),(1-c+b,\mu),(\beta,1);\left(a_j',\alpha_j'\right)_{1,P'};(1-a',1);\left(a_j,\alpha_j\right)_{1,P};(1-a,1);-\\
&S^{**}=(\alpha',1);(\alpha,1);\left(b_j',\beta_j'\right)_{1,Q'};(0,1);\left(b_j,\beta_j\right)_{1,Q};(0,1);(0,1)
\end{aligned}\right\} \tag{25}$$

and K*(T) can be obtained from K(T) by interchanging the parameters with dashes with those without dashes and following conditions are satisfied

$$\left.\begin{aligned}
&f(t)\in\Lambda\\
&\operatorname{Re}(\nu'+\nu+\zeta)>-2;\ \min_{1\le j\le M}\operatorname{Re}\left(\lambda'+\lambda_2'\frac{b_j'}{\beta_j'}+\lambda+\lambda_2\frac{b_j}{\beta_j}\right)>-2\\
&\operatorname{Re}(w_2)>0\ or\ \operatorname{Re}(w_2)=0\ and\ \operatorname{Re}\left[(\nu-w_1)\right]>0
\end{aligned}\right\} \tag{26}$$

Proof of (15), (19) & (22): To prove **Result 1**, we first express both the I-operators involved in its left hand side, in the integral form with the help of (10). Next we interchange the order of t-and x_1-integrals (which is permissible under the conditions stated), we easily have after a little simplification.

$$I_{x_2}^{\nu',\lambda'}\left[A_{n'},H',F_{p'};I_{x_1}^{\nu,\lambda}\left\{A_n,H,F_p;f(t)\right\}\right]=x_2^{-\nu'-\lambda'-1}\int_0^{x_2}t^{\nu}\Delta f(t)\,dt \tag{27}$$

$$\Delta=\int_t^{x_2}x_1^{\nu'-\nu-\lambda-1}(x_2-x_1)^{\lambda'}(x_1-t)^{\lambda}A_n\left[z_1\left(\frac{t}{x_1}\right)^{\nu_1}\left(1-\frac{t}{x_1}\right)^{\lambda_1}\right]A_{n'}\left[z_1'\left(\frac{x_1}{x_2}\right)^{\nu_1'}\left(1-\frac{x_1}{x_2}\right)^{\lambda_1'}\right]$$

Where

$$H_{P,Q}^{M,N}\left[z_2\left(1-\frac{t}{x_1}\right)^{\lambda_2}\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\ (b_j,\beta_j)_{1,Q}\end{matrix}\right.\right]H_{P',Q'}^{M',N'}\left[z_2'\left(1-\frac{x_1}{x_2}\right)^{\lambda_2'}\left|\begin{matrix}(a_j',\alpha_j')_{1,P'}\\ (b_j',\beta_j')_{1,Q'}\end{matrix}\right.\right]$$

$$F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(1-\frac{t}{x_1}\right)^{\lambda_3}\right]F_{p'}^{(\alpha',\beta';\tau',\mu')}\left[a',b';c';z_3'\left(1-\frac{x_1}{x_2}\right)^{\lambda_3'}\right]dx_1 \tag{28}$$

To evaluate Δ, we first express both the Fox's H-Functions and S-Generalized gauss hypergeometric functions in terms of their respective contour integral forms with the help of (1) and (6) respectively, next both the Appell polynomial are expressed in terms of the series with the help of (9). Further, weinterchange the order of summations and contour integral and get:

$$\Delta = \frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(a)\Gamma(a')B(b,c-b)B(b',c'-b')} \sum_{k=0}^{n} \sum_{k'=0}^{n'} \frac{a_{n-k} a_{n'-k'\Gamma(\nu+\nu'+\nu_1 k+\nu_1' k'+1)}}{k!k'!} z_1^{k} z_1'^{k'} t^{\nu_1 k}$$

$$\frac{1}{(2\pi w)^6} \int_{L_1} \dots \int_{L_6} \theta(\xi_2)\theta_1(\xi_2')\psi(\xi_1,\xi_3)\psi(\xi_1',\xi_3')(p^{-1})^{\xi_1}(p'^{-1})^{\xi_1'} z_2^{\xi_2} z_2'^{\xi_2'} (-z_3)^{\xi_3}(-z_3')^{\xi_3'} x_2^{-(\nu_1'+\lambda_1')k'-\lambda_2'\xi_2'-\lambda_3'\xi_3'} \quad (29)$$

$$\int_t^{x_2} x_1^{\nu'+\nu_1'k'-\nu-\lambda-(\nu_1+\lambda_1)k-\lambda_2\xi_2-\lambda_3\xi_3-1} (x_2-x_1)^{\lambda'+\lambda_1'k'+\lambda_2'\xi_2'+\lambda_3'\xi_3'} (x_2-x_1)^{\lambda+\lambda_1 k+\lambda_2\xi_2+\lambda_3\xi_3} dx_1$$

Now, we substitute $u = \left(\frac{x_2 - x_1}{x_2 - t}\right)$ in (29) and evaluate the u integral thus obtained with the help of known result [3].

Finally, re-interpreting the result in terms of the Multivariable H-function and substituting the values of Δ in (27), we get the right hand side of required result (15) after some simplification.

The results given by (19) and (22) can be proved on similar lines by making use of the results [3] and so we omit the details.

SPECIAL CASES OF COMPOSITION FORMULAE

As our composition formulae involve the Appell Polynomial, Fox H-function and S-Generalized Gauss's Hypergeometric Function, a large number of other composition formulae involving simpler functions and polynomials, can be obtained by specializing the functions involved in our composition formulae.

Thus if in these composition formulae, if we take $\lambda_2 = \lambda_2' = 0$ in (15), (19) and (22) H-Functions reduce to exponential function. Further, reducing exponential function to unity, reducing all the Appell Polynomials to unity and S-Generalized Gauss's Hypergeometric Function reducing to Gauss Hypergeometric Function by putting p = 0 thus obtained Gauss Hypergeometric Function reducing to unity, we get the corresponding expressions which are in essence the same as those given by Erdélyi [2].

Also, if we take p = 0 in (15), (19) and (22) S-Generalized Gauss's Hypergeometric Function reduce to Gauss Hypergeometric Function. Further reducing these Gauss Hypergeometric function to unity, reducing all the Appell Polynomials to unity and reducing all the H-functions to generalized hypergeometric functions, we get the corresponding expressions which are in essence the same as those given by Goyal and Jain [11].

CONCLUSION

We have obtained herein three composition formulae involving the Appell polynomial, Fox H-function and S-generalized Gauss hypergeometric function which are very general in nature. A large number of other composition formulae involving simpler functions and polynomials follow as simple special cases of our results.

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