Available online www.ejaet.com

European Journal of Advances in Engineering and Technology, 2016, 3(7): 34-37

Research Article ISSN: 2394 - 658X

# Aboodh Decomposition Method and its Application in Solving Linear and Nonlinear Heat Equations

**Rahmatullah Ibrahim Nuruddeen[1] and Aminu M Nass[2]**

*[1]Department of Mathematics, Federal University Dutse, Nigeria*
*[2]Department of Mathematics, University of Hafar Batin, Saudi Arabia*
*rahmatullah.n@fud.edu.ng*

**ABSTRACT**

*In this study, a recent added member of the integral transform family the 'Aboodh transform' is coupled with the Adomian decomposition method to solve linear and nonlinear heat conduction equations in what we called the Aboodh decomposition method. The Aboodh decomposition method works perfectly for both the linear and nonlinear heat conduction problems considered, and the results obtained are in full covenant with what is obtained in the literature.*

**Key words:** Aboodh Transform, Adomian Decomposition Method, Linear and Nonlinear Heat Equations

## INTRODUCTION

Heat conduction process has been a field of interest to engineers and many mathematicians for decades owing to its great applications industrially. Carslaw and Jaeger's book [1] discussed different heat conduction problems in different dimensions and situations. Nuruddeen and Zaman [2-3] solved heat conduction problems arising in circular cylinders. Polyanin *et al* [4] obtained the solutions of nonlinear steady and unsteady heat and mass-transfer problems using the method of generalized separation of variables. As a method, the Adomian decomposition method [5-6] is however found to be an efficient method for solving heat conduction problems in many situations. Ashfaque *et al* [7] used the Adomian decomposition method to determine the solution for nonlinear heat equations with temperature dependent thermal properties. Cheniguel and Ayadi [8] solved heat conduction problems via the use of Adomian decomposition method. Jebari *et al* [9] used the Adomian decomposition method to determine the solutions for nonlinear heat equations with exponential nonlinearities, while Ali [13] solved several boundary value problems with heat-like equations inclusive among others. Further, since the Adomian decomposition method [5-6] is found to be an efficient method for solving heat conduction problems, we now feel compelled to couple this same Adomian decomposition method with the recently introduced integral transform by Aboodh [10], the 'Aboodh transform' to handle linear and nonlinear heat conduction problems in what is known as Aboodh decomposition method. However, the compulsion is motivated by the successful use of the Aboodh transform to solve partial differential equations in [11] and also coupled with the homotopy perturbation method to solve fourth order parabolic partial differential equations with variable coefficients [12].

## ABOODH TRANSFORM

The Aboodh transform of the functions of exponential order belonging to a class A, where $A = \{u(t): \exists\, M, k_1, k_2 > 0\ s.t.\ |u(t)| < M\, e^{-vt}, \text{if } t \in (-1)^j \times [0,\infty)\}$ where u(t) is denoted by $A[u(t)] = U(v)$ and defined as

$$A\{u(t)\} = \frac{1}{v}\int_0^\infty e^{-vt}\,u(t)dt = U(v) \quad v \in (k_1, k_2). \tag{1}$$

Some properties of Aboodh transform are given in a table below

1. $A\{1\} = \frac{1}{v^2}$
2. $A\{t^n\} = \frac{n!}{v^{n+2}},\ n \geq 0$
3. $A\{\sin(at)\} = \frac{a}{v(v^2+a^2)}$
4. $A\{u^n(t)\} = v^n U(v) - \sum_{k=0}^{n-1} \frac{u^k(0)}{v^{2-n+k}}$

## DERIVATION OF THE METHOD

To present the Aboodh decomposition method, we consider the more general nonhomogeneous nonlinear partial differential equation

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = h(x,t) \tag{2}$$

with the initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x) \tag{3}$$

where $L$ is the second order linear differential operator with an inverse operator when existed as $L^{-1}(.) = \int_0^t \int_0^t (.)\, dtdt$, $R$ is the remaining linear operator less than $L$, $Nu(x,t)$ is the nonlinear operator and $h(x,t)$ is the nonhomogeneous term.

Now, on taking the Aboodh transform of equation (2) in '$t$' we get

$$A\{Lu(x,t)\} + A\{Ru(x,t)\} + A\{Nu(x,t)\} = A\{h(x,t)\}. \tag{4}$$

From the differentiation property of Aboodh transform, equation (4) becomes

$$v^2 A\{u(x,t)\} - u(x,0) - \frac{1}{v} u_t(x,0) + A\{Ru(x,t)\} + A\{Nu(x,t)\} = A\{h(x,t)\} \tag{5}$$

which can be simplified as

$$A\{u(x,t)\} - \frac{1}{v^2} u(x,0) - \frac{1}{v^3} u_t(x,0) + \frac{1}{v^2} A\{Ru(x,t)\} + \frac{1}{v^2} A\{Nu(x,t)\} = \frac{1}{v^2} A\{h(x,t)\} \tag{6}$$

Now, replacing the unknown function $u(x,t)$ by an infinite series of $u_m's$ i.e.,

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t), \tag{7}$$

and the nonlinear term by an infinite series of the Adomian polynomial $A_m's$ given by

$$Nu(x,t) = \sum_{m=0}^{\infty} A_m(u_0, u_1, u_2, \dots), \quad m = 0,1,2, \dots, \tag{8}$$

where

$$A_m = \frac{1}{m!} \frac{d^m}{d\lambda^m} [F(\sum_{i=0}^{\infty} \lambda^i u_i)]_{\lambda=0}, m = 0,1,2, \dots, \tag{9}$$

Substituting equations (7) and (8) into equation (6) we obtain

$$A\{\sum_{m=0}^{\infty} u_m(x,t)\} - \frac{1}{v^2} u(x,0) - \frac{1}{v^3} u_t(x,0) + \frac{1}{v^2} (A\{\sum_{m=0}^{\infty} Ru_m(x,t)\} + A\{\sum_{m=0}^{\infty} A_m(x,t)\}) = \frac{1}{v^2} A\{h(x,t)\} \tag{10}$$

Also, on substituting the initial conditions given in equation (3) we obtain

$$A\{\sum_{m=0}^{\infty} u_m(x,t)\} = \frac{1}{v^2} u(x,0) + \frac{1}{v^3} u_t(x,0) + \frac{1}{v^2} A\{h(x,t)\} - \frac{1}{v^2} (A\{\sum_{m=0}^{\infty} Ru_m(x,t)\} + A\{\sum_{m=0}^{\infty} A_m(x,t)\}) \tag{11}$$

Finally, on comparing both sides of equation (11) and thereafter taking the inverse Aboodh transform $A^{-1}$, we obtain the general solution given recursively by

$$\begin{cases} u_0(x,t) = f(x) + tg(x) + A^{-1}\left\{\frac{1}{v^2} A\{h(x,t)\}\right\}, & n = 0 \\ u_{n+1}(x,t) = -A^{-1}\left\{\frac{1}{v^2} A\{Ru_n(x,t) + A_n\}\right\}, & n \geq 0. \end{cases} \tag{12}$$

## APPLICATIONS OF THE METHOD

We consider some heat conduction problems comprising of linear, heat-like and nonlinear heat conduction problems in order to demonstrate the effectiveness of the Aboodh decomposition method as follows

**Example One**

Consider the one dimensional heat equation given by

$$u_t = u_{xx} \tag{13}$$

with the initial condition

$$u(x,0) = ae^{kx}, \tag{14}$$

where $a$ and $k$ are constants.

Now, on taking the Aboodh transform of both sides of equation (13), we obtain

$$A\{u_t\} = A\{u_{xx}\} \tag{15}$$

And on using the differentiation property we get

$$vA\{u(x,t)\} - \frac{1}{v} u(x,0) = A\{u_{xx}\}, \tag{16}$$

which gives

$$A\{u(x,t)\} = \frac{1}{v^2} u(x,0) + \frac{1}{v} A\{u_{xx}\} \tag{17}$$

Thus, on assuming the infinite series solution of the unknown function u(x, t) and then comparing both sides of equation, thereafter we take the inverse Aboodh transform; we get the general solution recursively as

$$\begin{cases} u_0(x,t) = u(x,0), & n = 0 \\ u_{n+1}(x,t) = A^{-1}\left\{\frac{1}{v} A\{u_{n_{xx}}\}\right\}, & n \geq 0. \end{cases} \tag{18}$$

We now determine the first few terms as follows

$$u_0(x,t) = ae^{kx}$$
$$u_1(x,t) = A^{-1}\left\{\frac{1}{v}A\{u_{0_{xx}}\}\right\}$$
$$= ak^2e^{kx}\,\mathrm{A}^{-1}\{\tfrac{1}{v^3}\}$$
$$= ak^2te^{kx},$$
$$u_2(x,t) = A^{-1}\left\{\frac{1}{v}A\{u_{1_{xx}}\}\right\}$$
$$= ak^4e^{kx}\mathrm{A}^{-1}\{\tfrac{1}{v^4}\}$$
$$= \frac{ak^4t^2e^{kx}}{2!},$$
$$u_3(x,t) = A^{-1}\left\{\frac{1}{v}A\{u_{2_{xx}}\}\right\}$$
$$= ak^6e^{kx}\mathrm{A}^{-1}\{\tfrac{1}{v^5}\}$$
$$= \frac{ak^6t^3e^{kx}}{3!},$$

and so on. Thus, on summing the above iterations up, the solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = ae^{kx} + ak^2te^{kx} + \frac{ak^4t^2e^{kx}}{2!} + \frac{ak^6t^3e^{kx}}{3!} + \cdots$$
$$= ae^{kx}\left(1 + (k^2t) + \frac{(k^2t)^2}{2!} + \frac{(k^2t)^3}{3!} + \cdots\right) = ae^{k(x+kt)}.$$

**Example Two**

Consider the heat-like equation

$$u_t = \frac{1}{2}x^2u_{xx} \tag{19}$$

with the initial condition

$$u(x,0) = x^2 \tag{20}$$

From the above procedure, equation (19) has the general solution recursively given by

$$\begin{cases} u_0(x,t) = u(x,0), & n = 0 \\ u_{n+1}(x,t) = A^{-1}\left\{\frac{1}{v}A\left\{\frac{1}{2}x^2u_{n_{xx}}\right\}\right\}, & n \geq 0. \end{cases} \tag{21}$$

Thus, we get the first few terms are determined as follows

$$u_0(x,t) = x^2$$
$$u_1(x,t) = A^{-1}\left\{\frac{1}{v}A\left\{\frac{1}{2}x^2u_{0_{xx}}\right\}\right\}$$
$$= \mathrm{A}^{-1}\{\tfrac{x^2}{v^3}\}$$
$$= x^2t,$$
$$u_2(x,t) = A^{-1}\left\{\frac{1}{v}A\left\{\frac{1}{2}x^2u_{1_{xx}}\right\}\right\}$$
$$= \mathrm{A}^{-1}\{\tfrac{x^2}{v^4}\}$$
$$= \frac{x^2t^2}{2!},$$
$$u_3(x,t) = A^{-1}\left\{\frac{1}{v}A\left\{\frac{1}{2}x^2u_{2_{xx}}\right\}\right\}$$
$$= \mathrm{A}^{-1}\{\tfrac{x^2}{v^5}\}$$
$$= \frac{x^2t^3}{3!},$$

and so on. Thus, on summing the above iterations we get

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x^2 + x^2t + \frac{x^2t^2}{2!} + \frac{x^2t^3}{3!} + \cdots$$
$$= x^2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) = x^2e^t.$$

**Example Three**

Consider a two dimensional nonlinear heat equation

$$u_t + uu_x = u_{xx} \tag{22}$$

with the initial condition

$$u(x,0) = 2x \tag{23}$$

From the procedure described above, equation (22) has the general solution

$$\begin{cases} u_0(x,t) = u(x,0), & n = 0 \\ u_{n+1}(x,t) = A^{-1}\left\{\frac{1}{v}A\{u_{n_{xx}} - A_n\}\right\}, & n \geq 0, \end{cases} \tag{24}$$

where $A_n's$ are the Adomian polynomials of the nonlinear term given some few terms as

$$A_0 = u_0u_{0_x}, \qquad A_1 = u_{0_x}u_1 + u_0u_{1_x}, \qquad A_2 = u_{0_x}u_2 + u_{1_x}u_1 + u_{2_x}u_0 \ldots$$ and so on.

Now, few terms of the solution are as follows

$$
\begin{aligned}
u_0(x,t) &= 2x \\
u_1(x,t) &= A^{-1}\left\{\frac{1}{v}A\left\{u_{0_{xx}} - A_0\right\}\right\} \\
&= \mathrm{A}^{-1}\{\tfrac{-4x}{v^3}\} \\
&= -4xt, \\
u_2(x,t) &= A^{-1}\left\{\frac{1}{v}A\left\{u_{1_{xx}} - A_1\right\}\right\} \\
&= \mathrm{A}^{-1}\{\tfrac{16x}{v^4}\} \\
&= \frac{16xt^2}{2!}, \\
u_3(x,t) &= A^{-1}\left\{\frac{1}{v}A\left\{u_{2_{xx}} - A_2\right\}\right\} \\
&= \mathrm{A}^{-1}\{\tfrac{-48x}{v^5}\} \\
&= \frac{-96xt^3}{3!},
\end{aligned}
$$

and so on. Summing the above iterations, we get the solution as

$$
\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) = 2x - 4xt + \frac{16xt^2}{2!} - \frac{96xt^3}{3!} + \cdots \\
&= 2x(1 + (-2t) + (-2t)^2 + (-2t)^3 + \cdots) \\
&= \frac{2x}{1+2t}.
\end{aligned}
$$

## CONCLUSION

In conclusion, the newly introduced integral transform to solve ordinary differential equations, partial differential equations and integral equations, the Aboodh transform is coupled with the celebrated decomposition method, the Adomian decomposition method in what we called the 'Aboodh decomposition method' to solve both linear and nonlinear heat conduction problems arising in science and engineering processes. The coupling works fine and yields remarkable exact solutions for the linear and nonlinear heat conduction problems considered, and the solutions obtained agree with the existing exact solutions that are available in the literature.

## REFERENCES


[1] H Carslaw H and JC Jaeger, *The Conduction of Heat in Solids*, Clarendon Press, Oxford, **1959**.

[2] RI Nuruddeen and FD Zaman, Heat Conduction of a Circular Hollow Cylinder Amidst Mixed Boundary Conditions, *International Journal of Scientific Engineering and Technology*, **2016**, 5(1),18-22.

[3] RI Nuruddeen and FD Zaman, Heat Conduction in a Circular Cylinder with General Mixed Boundary Conditions, *Journal of Multidisciplinary Engineering Science and Technology*, **2016**, 3(1), 3653-3658.

[4] A Polyanin, AI Zhurov and AV Vyaz'min, Exact Solutions of Nonlinear Heat- and Mass-Transfer Equations, *Theoretical Foundations of Chemical Engineering*, **2000**, 34(5), 403-415.

[5] G Adomian, Solution of Physical Problems by Decomposition, *Computers and Mathematics with Applications*, **1994**, 27 (9-10), 145-154.

[6] G Adomian, A Review of the Decomposition Method in Applied Mathematics, *Journal of Mathematical and Analysis Applications*, **1988**, 13(5), 501-544.

[7] HB Ashfaque, M Ghulam, MT Mustafa and FD Zaman, Adomian Decomposition Method for a Nonlinear Heat Equation with Temperature Dependent Thermal Properties, *Hindawi Mathematical Problems in Engineering*, **2009**, 2009(2009), 1-12.

[8] A Cheniguel and A Ayadi, Solving Heat Equation by the Adomian Decomposition Method, *Proceedings of the World Congress on Engineering*, London, UK, **2011**, 1.

[9] R Jebari, I Ghanmiand and A Boukricha, Adomian Decomposition Method for Solving Nonlinear Heat Equation with Exponential Nonlinearity, HIKARI Ltd, *International Journal of Mathematical Analysis*, **2013**, 7(15), 725-734.

[10] KS Aboodh, The New Integral Transform 'Aboodh Transform', *Global Journal of Pure and Applied Mathematics*, **2013**, 9(1), 35-43.

[11] KS Aboodh, Application of New Transform 'Aboodh Transform' to Partial Differential Equations, *Global Journal of Pure and Applied Mathematics*, **2014**, 10(2), 249-254.

[12] KS Aboodh, Solving Fourth Order Parabolic PDE with Variable Coefficients using Aboodh Transform Homotopy Perturbation Method, *Pure and Applied Mathematics Journal*, **2015**, 4(5), 219-224.

[13] EJ Ali, A New Technique of Initial Boundary Value Problems using Adomian Decomposition Method, *International Mathematical Forum*, **2012**, 7(17), 799-814.

[14] YAA. *et al*, Implementation of Lean Manufacturing: A Case Study at ASK Automotive Private Limited (India), *International Journal of Advanced Research in Science and Technology*, **2016**, 5 (1), 556-562.