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European Journal of Advances in Engineering and Technology, 2018, 5(6): 408-413

Research Article ISSN: 2394 - 658X

Thermoelastic problem for an infinitely long annular cylinder without energy dissipation (GN theory)

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ABSTRACT

In this paper we consider the problem of an infinitely long annular cylinder whose inner and outer surfaces are subjected to known surrounding temperatures and are traction free. The problem is in the context of the theory of thermoelasticity without energy dissipation. The Laplace transform with respect to time is used. The inversion process is carried out using a numerical method based on a Fourier series expansion. Numerical results are computed for the temperature, displacement and stress distributions. The numerical results are represented graphically. Comparison is made between the predictions here and those of the theory of thermoelasticity with one relaxation time.

Key words: annular cylinder - thermoelasticity without energy dissipation – Laplace transforms

INTRODUCTION

The classical theory of thermoelasticity has been generalized and modified into various thermoelastic models that run under the label of "hyperbolic thermoelasticity". The notation hyperbolic reflects the fact that thermal waves are modeled, avoiding the physical paradox of the infinite propagation speed of the classical model. At present, there are several theories of the hyperbolic thermoelasticity. The first was developed by Lord and Shulman [1] who obtained a wave-type heat equation by postulating a new law of heat conduction to replace the classical Fourier's law. This new law contains the heat flux vector as well as its time derivative. It also contains a new constant that acts as a relaxation time. The second was developed by Green and Lindsay [2]. This theory contains two constants that act as relaxation times and modifies all the equations of the coupled theory, not the heat conduction equation only. Both of these theories ensure finite speeds of propagation for heat wave. . Among the contributions to this theory are the works in [3-8].

The theory of thermoelasticity without energy dissipation (GN theory) was proposed by Green and Naghdi [9]. The most important aspect of this theory, which is not present in other thermoelasticity theories, is that this theory does not accommodate dissipation of thermal energy. Among the contributions to this theory are the works in [10-15].

FORMULATION OF THE PROBLEM

Let (r, φ, z) be cylindrical polar coordinates with the z-axis coinciding with the axis of an annular infinitely long elastic circular cylinder of a homogeneous, isotropic material of finite conductivity whose inner and outer radii are R_i, $i = 1,2$. The suffix 1 refers to the inner surface of the cylinder, while the suffix 2 refers to the outer surface. The surfaces of the cylinder are taken to be traction-free and are in contact with media of known temperatures.

Due to the physics of the problem, all the functions considered will depend on r and t only. The displacement components have the form

$\underline{\mathbf{u}} = (u(r, t), 0, 0)$.

The basic equations due to Green and Nagdhi [9] in the absence of body forces and heat sources for isotropic elastic medium are given by:

Equation of motion has the form

$$\mu\nabla^2 u + (\lambda + \mu)\, grad\, div\, u - \gamma\, grad\, T = \rho\ddot{u} \tag{1}$$

The generalized equation of heat conduction are given by

$$k^*\nabla^2 T = \rho c_E \frac{\partial^2 T}{\partial t^2} + \gamma T_0 \frac{\partial^2}{\partial t^2}(div\, u) \tag{2}$$

THE CONSTITUTIVE EQUATION

$$\sigma_{rr} = (\lambda + 2\mu)\frac{\partial u}{\partial r} + \lambda \frac{u}{r} - \gamma (T - T_0) \tag{3}$$

where λ and μ are Lamé's modulii, T is the absolute temperature of the medium, and γ is a material constant given by $\gamma = (3\lambda + 2\mu)\alpha_t$ where α_t is the coefficient of linear thermal expansion, C_E is the specific heat at constant strain, ρ is the density and T_0 is a reference temperature assumed to be such that $\left|(T-T_0)/T_0\right| << 1$. k^* is a material constant, characteristic of the theory.

Let us introduce the following non-dimension variables

$$r^* = c_1 \eta r,\ u^* = c_1 \eta u,\ ,t^* = c_1^2 \eta t,\ \sigma^* = \frac{\sigma}{\mu}\ ,\ \theta = \frac{\gamma (T - T_0)}{(\lambda + 2\mu)}.$$

The governing equations (1)-(3) in non-dimensional form become (dropping the asterisks for convenience)

$$\nabla^2 u + (\beta^2 - 1)\, grad\, e - \beta^2\, grad\, \theta = \beta^2 \ddot{u} \tag{4}$$

$$C_t^2 \nabla^2 \theta = \frac{\partial^2 \theta}{\partial t^2} + \varepsilon \frac{\partial^2 e}{\partial t^2} \tag{5}$$

$$\sigma_{rr} = \beta^2 \frac{\partial u}{\partial r} + (\beta^2 - 2)\frac{u}{r} - \beta^2 \theta \tag{6}$$

Where

$$C_t^2 = \frac{k^*}{\rho c_E c_1^2}\ ,\ \varepsilon = \frac{\gamma^2 T_0}{\rho c_E (\lambda + 2\mu)},\ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r},\ e = div\, u = \frac{\partial u}{\partial r} + \frac{u}{r}$$

Introducing the thermoelastic potential function Ψ, defined by $u = \frac{\partial \psi}{\partial r}$

Equations (4)-(6) given by

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\psi = \theta \tag{7}$$

$$C_t^2 \nabla^2 \theta = \frac{\partial^2 \theta}{\partial t^2} + \varepsilon \frac{\partial^2}{\partial t^2}\nabla^2 \psi \tag{8}$$

$$\sigma_{rr} = \beta^2 \frac{\partial^2 \psi}{\partial r^2} + (\beta^2 - 2)\frac{1}{r}\frac{\partial \psi}{\partial r} - \beta^2 \theta \tag{9}$$

SOLUTION OF THE PROBLEM IN THE LAPLACE TRANSFORM DOMAIN

Applying the Laplace transform with parameter s defined by the relation

$$\bar{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt\,,$$

to both sides of equations (4)-(6), we obtain

$$\left(\nabla^2 - s^2\right)\ \overline{\psi} = \overline{\theta} \tag{10}$$

$$C_t^2 \nabla^2 \overline{\theta} = s^2 \overline{\theta} + \varepsilon s^2 \nabla^2 \overline{\psi} \tag{11}$$

$$\overline{\sigma}_{rr} = \beta^2 \frac{\partial^2 \overline{\psi}}{\partial r^2} + (\beta^2 - 2)\frac{1}{r}\frac{\partial \overline{\psi}}{\partial r} - \beta^2 \overline{\theta} \tag{12}$$

Eliminating $\overline{\theta}$ from equations (10) and (11), we get

$$\left\{\nabla^4 - \frac{(C_t^2+1+\varepsilon)}{C_t^2} s^2 \nabla^2 + \frac{s^4}{C_t^2}\right\}\overline{\psi} = 0 \tag{13}$$

Equation (13) can be factorized as

$$\left(\nabla^2 - k_1^2\right)\left(\nabla^2 - k_2^2\right)\overline{\theta} = 0 \tag{14}$$

where k_1^2 and k_2^2 are the roots of the characteristic equation

$$k^4 - \frac{(C_t^2+1+\varepsilon)}{C_t^2} s^2 k^2 + \frac{s^4}{C_t^2} = 0$$

Since the operators in (14) are permutable it follows from the theory of linear differential equations that ψ is the linear combination of the two solutions of

$$\left(\nabla^2 - k_i^2\right)\overline{\psi} = 0$$

Hence

$$\overline{\psi} = \sum_{i=1}^{2} \left[A_i I_0(k_i r) + B_i K_0(k_i r)\right] \tag{15}$$

where A_i and B_i, $i = 1, 2$ are parameters depending on s to be determined from the boundary conditions and I_0, K_0 are the modified Bessel functions of the first and second kinds respectively.

Substituting from (15) into (l0), we get

$$\overline{\theta} = \sum_{i=1}^{2} \left[A_i\left(k_i^2 - s^2\right)I_0(k_i r) + B_i\left(k_i^2 - s^2\right)K_0(k_i r)\right] \tag{16}$$

the displacement $\overline{u}$ is given by

$$\overline{u} = \sum_{i=1}^{2} \left[A_i\, k_i I_1(k_i r) - B_i\, k_i K_1(k_i r)\right] \tag{17}$$

the stress tensor are obtained from equations (12), (15) and (16), they have the form

$$\overline{\sigma}_{rr} = \sum_{i=1}^{2} \left\{A_i\left[\beta^2 s^2 I_0(k_i r) - \frac{2k_i}{r} I_1(k_i r)\right] + B_i\left[\beta^2 s^2 K_0(k_i r) + \frac{2k_i}{r} K_1(k_i r)\right]\right\} \tag{18}$$

The boundary conditions of the problem can be written as:

$$\sigma_{rr} = 0 \;,\; T = 0 \;,\quad at\; r = R_1$$

$$\sigma_{rr} = 0 \;,\; T = f(t) = \theta_0 H(t) \quad ,\quad at\; r = R_2$$

where H(t) is the Heaviside unit step function. Taking the Laplace transform of both sides of the preceding equations, we obtain

$$\begin{aligned} &\overline{\sigma}_{rr} = 0 \;,\; \overline{\theta} = 0 \;,\quad at\; r = R_1 \\ &\overline{\sigma}_{rr} = 0 \;,\; \overline{\theta} = \frac{\theta_0}{s} \quad ,\quad at\; r = R_2 \end{aligned} \tag{19}$$

Using the boundary conditions (19), we get the following linear system of equations in the 4 parameters A_i and B_i, $i = 1, 2$

$$\sum_{i=1}^{2} \left[A_i\left(k_i^2 - s^2\right)I_0(k_i R_1) + B_i\left(k_i^2 - s^2\right)K_0(k_i R_1)\right] = 0 \tag{20}$$

$$\sum_{i=1}^{2} \left\{A_i\left[\beta^2 s^2 I_0(k_i R_1) - \frac{2k_i}{R_1} I_1(k_i R_1)\right] + B_i\left[\beta^2 s^2 K_0(k_i R_1) + \frac{2k_i}{R_1} K_1(k_i R_1)\right]\right\} = 0 \tag{21}$$

$$\sum_{i=1}^{2} \left[A_i\left(k_i^2 - s^2\right)I_0(k_i R_2) + B_i\left(k_i^2 - s^2\right)K_0(k_i R_2)\right] = \frac{\theta_0}{s} \tag{22}$$

$$\sum_{i=1}^{2} \left\{A_i\left[\beta^2 s^2 I_0(k_i R_2) - \frac{2k_i}{R_2} I_1(k_i R_2)\right] + B_i\left[\beta^2 s^2 K_0(k_i R_2) + \frac{2k_i}{R_2} K_1(k_i R_2)\right]\right\} = 0 \tag{23}$$

Numerical inversion of the Laplace transforms

We shall now outline the method used to invert the Laplace transforms in the above equations. Let $\bar{f}(r,s)$ be the Laplace transform of a function *f(r, t)*. The inversion formula for Laplace transforms can be written as [16]

$$f(\mathbf{r},t)=\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}\,\bar{f}(\mathbf{r},s)\,ds\ ,$$

where *d* is an arbitrary real number greater than all the real parts of the singularities of $\bar{f}(\mathbf{r},s)$. Taking $s = d + i\,y$, the above integral takes the form

$$f(\mathbf{r},t)=\frac{e^{dt}}{2\pi}\int_{-\infty}^{\infty} e^{ity}\,\bar{f}(\mathbf{r},d+iy)\,dy\ .$$

Expanding the function *h(r, t) = exp(-dt) f(r, t)* in a Fourier series in the interval [0,2L], we obtain the approximate formula [17]

$$f(r,t) = f_\infty(r,t) + E_D\ ,$$

where

$$f_\infty(\mathbf{r},t)=½\,c_0+\sum_{k=1}^{\infty} c_k \quad ,\ for\ 0\le t\le 2L\ , \tag{24}$$

and

$$c_k=\frac{e^{dt}}{L}\,Re\left[\,e^{ik\pi t/L}\,\bar{f}(\mathbf{r},d+ik\pi/L)\right]. \tag{25}$$

The discretization error, E_D, can be made arbitrarily small by choosing *d* large enough [17].

As the infinite series in (24) can only be summed up to a finite number *N* of terms, the approximate value of *f(r, t)* becomes

$$f_N(\mathbf{r},t)=½\,c_0+\sum_{k=1}^{N} c_k \quad ,\ for\ 0\le t\le 2L\ . \tag{26}$$

Using the above formula to evaluate *f(r,t)*, we introduce a truncation error E_T that must be added to the discrimination error to produce the total approximation error.

Two methods are used to reduce the total error. First, the `Korrecktur` method is used to reduce the discrimination error. Next, the ε-algorithm is used to reduce the truncation error and therefore to accelerate convergence.

The Korrecktur-method uses the following formula to evaluate the function *f(r, t)*

$$f(r,t) = f_\infty(r,t) - e^{-2dL} f_\infty(r, 2L+t) + E`_D\ ,$$

where the discrimination error $|E'_D| << |E_D|$ [17].

Thus, the approximate value of *f(r, t)* becomes

$$f_{NK}(r,t) = f_N(r,t) - e^{-2dL} f_{N`}(r, 2L+t)\ . \tag{27}$$

N` is an integer such that $N` < N$.

We shall now describe the ε-algorithm that is used to accelerate the convergence of the series in (24). Let *N* be an odd natural number and let

$$s_m=\sum_{k=1}^{m} c_k\ ,$$

be the sequence of partial sums of (24). We define the ε-sequence by

$$\varepsilon_{0,m}=0\,,\varepsilon_{1,m}=s_m\quad ,\quad m=1,2,3,\ \dots\ .$$

And $\varepsilon_{n+1,m}=\varepsilon_{n-1,m+1}+\dfrac{1}{\varepsilon_{n,m+1}-\varepsilon_{n,m}}\quad ,\ n,m=1,2,3,\dots$

It can be shown that [17] the sequence

$$\varepsilon_{1,1}\ ,\ \varepsilon_{3,1}\ ,\ .\ .\ .\ ,\ \varepsilon_{N,1,\dots}$$

Converges to $f(r,t) + E_D - c_0/2$ faster than the sequence of partial sums

$$s_m\quad ,\quad m=1,2,3,\ \dots\ .$$

The actual procedure used to invert the Laplace Transforms consists of using equation (27) together with the ε-algorithm. The values of *d* and *L* are chosen according the criteria outlined in [17].

Numerical results

For fixed values of r and t, the linear system (20)-(23) is solved numerically for the unknowns A_i, B_i, i = 1, 2, and the results are substituted into (16), (17) and (18) to obtain the value of θ, u and σ_{rr}. The functions are evaluated for value of time namely t = 0.1 at points inside the cylindrical annuals. These results are shown in Figs l-3 these figures include also the graphs of the corresponding functions for the theory of thermoelasticity without energy dissipation (GN theory). For numerical computations we have used the copper material with

$\varepsilon = 0.0168$, $\beta^2 = 3.342$, $R_I = 1$, $R_2 = 2$, $C_t^2 = 40$

LS theory is represented by dotted lines while GN theory is represented by solid lines.

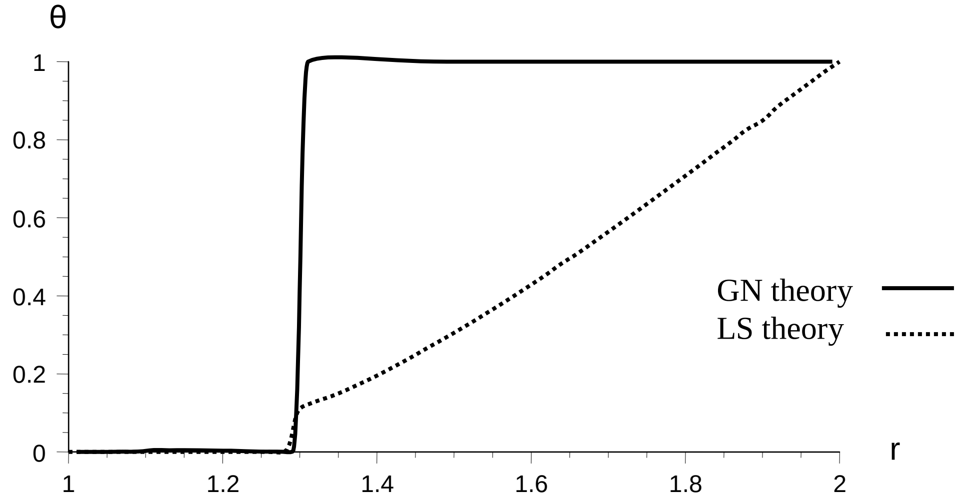

Fig. 1 Temperature Distribution

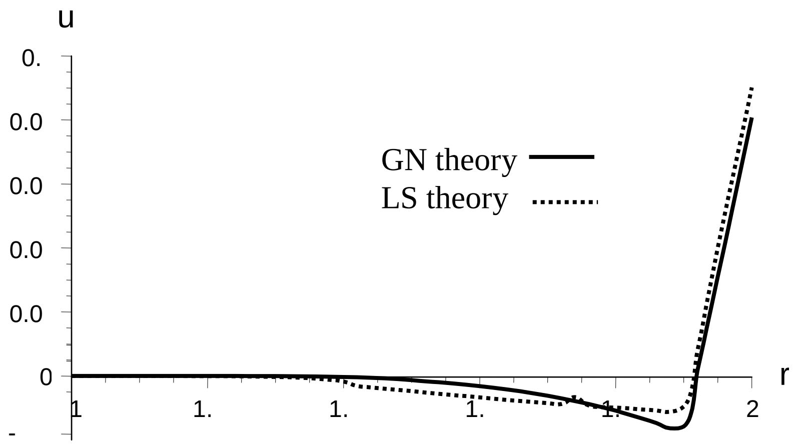

Fig. 2 Displacement Distribution

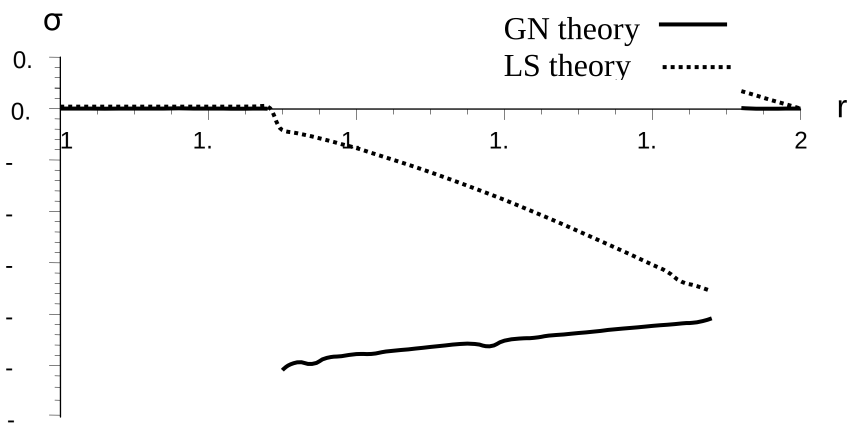

Fig. 3 Stresses Distribution

Concluding Remarks

In this work, we have discussed the model of thermoelastic problem for an infinitely long annular cylinder without energy dissipation (GN theory) the context of Green and Naghdi theory without energy dissipation. The problem is solved by means of the Laplace transform and Laplace inversion. We concluded that:

It is clear from the above figures that results for generalized thermoelasticity (LN theory) are distinctly different from those of theory of thermoelasticity without energy dissipation (GN theory) for small values of time. The temperature and stress distributions have two finite jumps while the displacement is continuous everywhere having a discontinuous first derivative. The first discontinuity does not show in figures 1 because it is very small.

The difference between the predictions of the theories of LS and GN is most apparent in the graph of the temperature distribution. In the LS theory the temperature decreases monotonically signifying continuous dissipation of heat energy. This is not the case for GN theory.

The mechanical distributions indicate that the wave propagates as a wave with finite velocity in medium in two cases. It is completely different from the case for the classical theory of thermoelasticity where an infinite speed of propagation is inherent and hence all the considered functions have a non-zero value for any point in the medium.

The fact that in thermoelasticity without energy dissipation, the waves propagate with finite speeds is evident in all these figures.

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